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Baryon masses, chiral extrapolations, and all that*

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Abstract. We calculate the baryon octet masses to fourth order in chiral perturbation theory employing dimensional and cut-off regularization. We analyze the pion and kaon mass dependences of the baryon masses based on the MILC data. We show that chiral perturbation theory gives stable chiral extrapolation functions for pion (kaon) masses below 550 (600) MeV. For the pion-nucleon sigma term we find $\sigma_{\pi N}(0) = 39.5\ldots 46.7\,\mathrm{MeV}$.

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1 Introduction and summary

The masses of the ground-state baryon octet are of fundamental importance in the investigation of three-quark states in QCD. With the advent of improved techniques in lattice QCD and systematic studies within the framework of chiral perturbation theory, one can hope to gain an understanding of these quantities from first principles. Present-day lattice calculations are done at unphysical quark masses above the physical values, therefore chiral extrapolations are needed to connect lattice results with the physical world, provided that the masses are not too high (for an early approach to this problem, see, *e.g.*, [1]). With the recent-low-quark-mass data from the MILC Collaboration [2,3] it appears to be possible to apply chiral extrapolation functions derived from chiral perturbation theory (CHPT) to the masses of the ground-state baryons (for a review on CHPT, see [4]). In this paper, we analyze the baryon masses as a function of the pion and the kaon masses in CHPT based on the MILC data. Other pertinent lattice papers are, *e.g.*, [5–9]. It should be said from the beginning that we ignore the effects of a) the finite volume, b) the finite lattice spacing and c) the staggered approximation in this study[1] since our aim is more modest —we want to find out whether these chiral extrapolations can be used for the presently available lattice data. Once this test is performed, one should then apply the full formalism including the above-mentioned effects. Note that the quark mass expansion of QCD is turned into an expansion in Goldstone boson (GB) masses in CHPT —we thus use both terms synonymously.

The baryon masses have been analyzed in various versions of baryon CHPT to third and fourth order; for an incomplete list of references see [13–19]. Our investigation extends the work in the two-flavor sector presented in [20] and we heavily borrow from the earlier $SU(3)$ calculations of [15,17]. The pertinent results of this investigation can be summarized as follows:

1) We have calculated the baryon masses to third and fourth order in the chiral expansion making use of cut-off regularization as proposed in [20]. As in that paper, we have also considered an improvement term at third order to cancel the leading cut-off dependence in the baryon masses, see sect. 3.2.
2) The improvement term consists of three independent terms, whose cut-off–independent coefficients have been determined by considering the nucleon mass (to allow for a direct comparison with the $SU(2)$ calculation of ref. [20]). We have demanded that for the physical pion and the physical kaon mass the nucleon mass passes through its physical value. This fixes two parameter combinations. The third parameter is determined from a best fit to the trend of the earlier MILC data [2] for $m_N(M_\pi)$, cf. fig. 2, under the condition that the deviation from the earlier determination of the corresponding low-energy constants in [15] is of natural size. We find indeed a visible improvement in

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[1] Some of these effects are studied in [10–12] (and references therein). For our purpose, it is sufficient to treat the MILC data as interpolated to the continuum limit. We are grateful to C. Bernard for comments on this issue.

the description of the lattice data and also much better stability under variations of the cut-off.

3) The full fourth-order calculation utilizing the low-energy constants as determined from the improvement term leads to an accurate description of the MILC data for pion masses below 550 MeV, see again fig. 2. Note that the two lowest-mass points of the more recent MILC data [3] cannot be described. Also, the use of the low-energy constants (LECs) from [15] gives a less satisfactory description. We have discussed the theoretical uncertainty of this procedure, cf. fig. 3.
4) From the quark mass dependence of the nucleon mass, we can deduce the pion-nucleon sigma term. For the best sets of low-energy constants, we find $\sigma_{\pi N}(0) = 39.5\ldots46.7\,\mathrm{MeV}$.
5) The kaon mass dependence of the nucleon mass is less well determined. Still, the extrapolation functions can be applied to kaon masses below $\simeq 600\,\mathrm{MeV}$. We deduce that the baryon octet mass in the chiral limit lies in the interval $710\,\mathrm{MeV} \lesssim m_0 \lesssim 1070\,\mathrm{MeV}$. This is consistent with earlier estimates, see, *e.g.*, [15].
6) We have also considered the pion and kaon mass dependences of the Λ, the Σ and the Ξ and compared to the existing MILC data, cf. figs. 6-11. Note that we have not fitted to these masses. Our chiral extrapolations for the Σ and in particular for the Ξ as a function of the pion mass are flatter than the MILC data. This is partly due to our strategy of fixing all parameters on the nucleon mass. We remark, however, that one should expect a decreased pion mass dependence as the number of strange valence quarks increases.

The material in this paper is organized as follows. Section 2 contains the effective Lagrangian and a short discussion about the various regularization methods employed in our calculations. In sect. 3, the ground-state baryon octet masses are given at third, third improved and fourth order in the chiral expansion. In particular, we concentrate on the differences to the $SU(2)$ case [20] (we refer to that paper for many details). Our results for the various baryon masses as functions of the pion and the kaon masses and the stability of these results under cut-off variations are given and discussed in sect. 4. Many technicalities are relegated to the appendices.

2 Formalism I: generalities

In this section, we display the effective Lagrangian underlying our calculations and discuss briefly the cut-off regularization utilized and its relation to the more standard dimensional regularization (DR). We borrow heavily from the work presented in refs. [15,20] and refer the reader for more details to these papers.

2.1 Effective Lagrangian

Our calculations are based on an effective chiral meson-baryon Lagrangian in the presence of external sources (like *e.g.*, photons) supplemented by a power counting in terms of quark (meson) masses and small external momenta. Its generic form consists of a string of terms with increasing chiral dimension,

$$\mathcal{L} = \mathcal{L}_{\phi B}^{(1)} + \mathcal{L}_{\phi B}^{(2)} + \mathcal{L}_{\phi B}^{(3)} + \mathcal{L}_{\phi B}^{(4)} + \mathcal{L}_{\phi}^{(2)} + \mathcal{L}_{\phi}^{(4)} \,. \tag{2.1}$$

Here, B collects the baryon octet and ϕ stands for the Goldstone boson octet. The superscript denotes the power in the genuine small parameter q (denoting Goldstone boson masses and/or external momenta). The explicit representations of ϕ and B are

$$\phi(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

$$B(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}. \tag{2.2}$$

A complete one-loop (fourth-order) calculation must include all tree level graphs with insertions from all terms given in eq. (2.1) and loop graphs with at most one insertion from $\mathcal{L}_{\phi B}^{(2)}$ or $\mathcal{L}_{\phi}^{(2)}$. Throughout, we employ the heavy-baryon approach, which allows for a consistent power counting since the large mass scale (the baryon mass m_B) is transformed from the propagator into a string of $1/m_B$ suppressed interaction vertices. The lowest-order (dimension one) effective Lagrangian takes the canonical form

$$\begin{aligned}\mathcal{L}_{\phi B}^{(1)} &= i\,\mathrm{Tr}(\bar{B}[v\cdot D, B]) \\ &\quad + F\,\mathrm{Tr}(\bar{B}S_\mu[u^\mu, B]) + D\,\mathrm{Tr}(\bar{B}S_\mu\{u^\mu, B\})\,,\end{aligned} \tag{2.3}$$

where v_μ is the four-velocity of the baryon subject to the constraint $v^2 = 1$, Tr denotes the trace in flavor space and D and F are the leading axial-vector couplings, $D \simeq 3/4$ and $F \simeq 1/2$. Furthermore, S_μ is the spin-vector and $u_\mu = iu^\dagger\nabla_\mu U u^\dagger$, where $U = u^2$ collects the Goldstone bosons (for more details, see [4]). For the calculation of the self-energy (mass), it suffices to use the partial derivative ∂_μ instead of the chiral covariant derivative D_μ. The dimension-two chiral Lagrangian can be decomposed as (for details see [15] and [17])

$$\mathcal{L}_{\phi B}^{(2)} = \mathcal{L}_{\phi B}^{(2,\mathrm{br})} + \sum_{i=1}^{19} b_i O_i^{(2)} + \mathcal{L}_{\phi B}^{(2,\mathrm{rc})} \tag{2.4}$$

with

$$\begin{aligned}\mathcal{L}_{\phi B}^{(2,\mathrm{br})} &= b_D\mathrm{Tr}[\bar{B}\{\chi_+, B\}] + b_F\mathrm{Tr}[\bar{B}[\chi_+, B]] \\ &\quad + b_0\mathrm{Tr}[\bar{B}B]\mathrm{Tr}[\chi_+],\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sum_{i=1}^{19} b_i O_i^{(2)} =\; & b_1 \mathrm{Tr}[\bar{B}[u_\mu,[u^\mu,B]]] + b_2 \mathrm{Tr}[\bar{B}[u_\mu,\{u^\mu,B\}]] \\ & + b_3 \mathrm{Tr}[\bar{B}\{u_\mu,\{u^\mu,B\}\}] \\ & + \left((b_4 - m_0 b_{15}) m_0 + \frac{1}{4}(b_{12} - m_0 b_{18}) m_0\right) \\ & \times \mathrm{Tr}[\bar{B}[v\cdot u,[v\cdot u,B]]] \\ & + (b_5 + b_6 - m_0 b_{16}) m_0 \mathrm{Tr}[\bar{B}[v\cdot u,\{v\cdot u,B\}]] \\ & + \left((b_7 - m_0 b_{17}) m_0 + \frac{3}{4}(b_{12} - m_0 b_{18})\right) \\ & \times m_0 \mathrm{Tr}[\bar{B}\{v\cdot u,\{v\cdot u,B\}\}] \\ & + b_8 \mathrm{Tr}[\bar{B}B]\mathrm{Tr}[u^\mu u_\mu] \\ & + b_{11} 2i\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[\bar{B}u_\mu] v_\alpha S_\beta \mathrm{Tr}[u_\nu B] \\ & + \left((b_9 - m_0 b_{19}) m_0 - \frac{1}{2}(b_{12} - m_0 b_{18}) m_0\right) \\ & \times \mathrm{Tr}[\bar{B}B]\mathrm{Tr}[v\cdot u\, v\cdot u] \\ & + b_{13} 2i\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[\bar{B} v_\alpha S_\beta [[u_\mu,u_\nu],B]] \\ & + b_{14} 2i\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}[\bar{B} v_\alpha S_\beta \{[u_\mu,u_\nu],B\}] , \qquad (2.6)\end{aligned}$$

with m_0 the octet baryon mass in the chiral limit[2]. Explicit symmetry breaking embodied in the external source $\chi_+ \sim \mathcal{M}$ (with $\mathcal{M}$ the diagonal quark mass matrix) only starts at this order, collected in $\mathcal{L}_{\phi B}^{(2,\mathrm{br})}$. It is parameterized in terms of the LECs b_0, b_D and b_F. Throughout, we work in the isospin limit $m_u = m_d = \hat{m}$ and thus consider four different baryon states, the nucleon doublet (N), the lambda (Λ), the sigma triplet (Σ) and the cascade doublet (Ξ) (isospin breaking corrections are discussed in [17] and [21]). The operators with the LECs b_i ($i = 1, \ldots, 19$) (more precisely: eight combinations of these) only appear as insertions in fourth-order tadpole graphs (see [17] for a detailed discussion). Note that in [15] the contributions from various combinations of dimension-two LECs were effectively subsumed in one corresponding coupling since operators $\sim k^2$ and $\sim (v\cdot k)^2$ lead to the same contribution to the baryon masses. We can only do that at a later stage in the calculation so as to be able to consistently work out the renormalization of these dimension-two operators. As will become clear later, while in DR the b_i are finite numbers, this is not the case if one employs cut-off regularization. We stress that in the end, our chiral extrapolation functions will depend on the three symmetry-breaking LECs (b_0, b_D, b_F) and four combinations of two-derivative LECs $\sim k^2$ since, as shown in [15, 17], the two-derivative LECs $\sim (v\cdot k)^2$ can be absorbed entirely in the other four dimension-two LECs $\sim k^2$ and some of the fourth-order LECs. Furthermore, these independent two-derivative LECs are precisely the ones that were already determined in [15] and we can use these values, after transformation from the dimensional regularization representation to the one employed here, as detailed in sect. 3. Finally, we remark that the recoil terms collected in $\mathcal{L}_{\phi B}^{(2,\mathrm{rc})}$ are given in [15]. Similarly, there are further recoil corrections $\sim 1/m_B^2$ collected in $\mathcal{L}_{\phi B}^{(3,\mathrm{rc})}$. These involve no unknown parameters, their explicit form is also given in [15]. To end this section, we give the fourth-order terms relevant for our calculations,

$$\begin{aligned}\mathcal{L}_{\phi B}^{(4)} =\; & d_1 \mathrm{Tr}(\bar{B}[\chi_+,[\chi_+,B]]) + d_2 \mathrm{Tr}(\bar{B}[\chi_+,\{\chi_+,B\}]) \\ & + d_3 \mathrm{Tr}(\bar{B}\{\chi_+,\{\chi_+,B\}\}) + d_4 \mathrm{Tr}(\bar{B}\chi_+)\mathrm{Tr}(\chi_+ B) \\ & + d_5 \mathrm{Tr}(\bar{B}[\chi_+,B])\mathrm{Tr}(\chi_+) + d_7 \mathrm{Tr}(\bar{B}B)\mathrm{Tr}(\chi_+)\mathrm{Tr}(\chi_+) \\ & + d_8 \mathrm{Tr}(\bar{B}B)\mathrm{Tr}(\chi_+^2) . \qquad (2.7)\end{aligned}$$

We remark that we have employed the notation of [15] to facilitate the comparison with that work and also use some of the LECs determined there.

2.2 Regularization schemes

We briefly recall the salient features of the various regularization schemes employed in calculating the baryon masses. Heavy-baryon CHPT together with DR was used, *e.g.*, in the early papers [13–15], and a fourth-order calculation using infrared regularization (which is also based on DR to deal with the UV divergences in the loop graphs) was reported in [17] (for an earlier incomplete calculation, see [16] and a recent calculation in the extended on-mass shell renormalization scheme was reported in [19]). To be definite, consider the leading one-loop pion graph for the nucleon mass (the sunset diagram with insertions from the leading-order Lagrangian which is of third order). In the heavy-baryon approach, it is given by

$$\begin{aligned} I_N^\pi &= \frac{c}{4} J(0)\, M_\pi^2 , \\ J(0) &= \frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(M_\pi^2 - k^2 - i\epsilon)(v\cdot k - i\epsilon)} , \qquad (2.8)\end{aligned}$$

with $c = (D+F)^2/F_0^2$ and F_0 the pseudoscalar decay constant in the chiral limit. In DR, the loop function $J(0)$ is finite and can be expressed as

$$J(0) = M_\pi^{d-3}(4\pi)^{-d/2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3-d}{2}\right) = -\frac{M_\pi}{8\pi} , \qquad (2.9)$$

with d the number of space-time dimensions and we have set $d = 4$ on the right-hand side of eq. (2.9). This gives the time-honored leading non-analytic contribution

$$I_N^\pi = -\frac{c}{32\pi} M_\pi^3 . \qquad (2.10)$$

Note that in DR no power law divergences appear and therefore loop graphs cannot renormalize the baryon mass

[2] To differentiate between the chiral-limit mass and the physical mass is completely standard in chiral perturbation theory. The parameters entering the chiral Lagrangian are to be taken in the chiral limit (like here the baryon mass m_0), and only when one calculates observables, one expresses the results in terms of physical parameters (like here the m_B). These only differ by terms proportional to the quark masses. Since we are after the chiral expansion of the baryon masses, we *must* operate with the chiral-limit mass.

in the chiral limit and the dimension-two LECs which leads to self-energy terms $\sim M_\pi^2$. If we instead use a three-momentum cut-off as suggested in [20], the same diagram leads to the expression

$$I_N^\pi = -\frac{c}{16\pi^2}\int_0^\infty \mathrm{d}y \int_0^\Lambda \mathrm{d}|\boldsymbol{k}| \frac{\boldsymbol{k}^4}{(\boldsymbol{k}^2+M_\pi^2+y^2)^{3/2}} = -\frac{c}{16\pi^2}\left\{\frac{\Lambda^3}{3} - M_\pi^2\Lambda - M_\pi^3 \arctan\frac{M_\pi}{\Lambda}\right\} - \frac{c}{32\pi}M_\pi^3\ . \quad (2.11)$$

Note that besides the contribution $\sim M_\pi^3$ that is free of the cut-off, we have additional divergent and finite terms. The cubic divergence independent of the pion mass leads to a renormalization of the baryon mass in the chiral limit, whereas the term $\sim M_\pi^2\Lambda$ renormalizes the dimension-two LECs (the precise relations between the bare and the renormalized parameters will be given in the following section). Also, the finite arctan term is formally of higher order since it only starts to contribute at order M_π^4. Chiral symmetry has been manifestly maintained by this procedure since no structures besides the ones already appearing in the effective Lagrangian are needed to absorb all divergences (see also [20] and the more systematic work reported in [22]). Also, the DR result can be formally obtained when letting the cut-off tend to infinity. This will also be discussed in more detail below.

3 Formalism II: baryon masses

This section contains the basic formalism to calculate the baryon masses to fourth order in the chiral expansion. We briefly discuss the third-order result and the introduction of an improvement term as proposed in [20]. We then proceed to present the central new results, namely the baryon masses to fourth order utilizing cut-off regularization. The calculation of the baryon self-energy and the corresponding mass shift at a given order in the chiral expansion is briefly outlined in appendix A.

3.1 Baryon masses at third order

The calculation of the baryon masses to third order in cut-off regularization is straightforward. Utilizing the loop integrals collected in appendix B, one obtains

$$\begin{aligned} m_B = {} & m_0^{(r)} + \gamma_B^D b_D^{(r)} + \gamma_B^F b_F^{(r)} - 2b_0^{(r)}(M_\pi^2+2M_K^2) \\ & - \frac{1}{24\pi F_0^2}\left[\alpha_B^\pi M_\pi^3 + \alpha_B^K M_K^3 + \alpha_B^\eta M_\eta^3\right] \\ & + \frac{1}{12\pi^2 F_0^2}\Bigg[\alpha_B^\pi M_\pi^3 \arctan\frac{M_\pi}{\Lambda} \\ & + \alpha_B^K M_K^3 \arctan\frac{M_K}{\Lambda} + \alpha_B^\eta M_\eta^3 \arctan\frac{M_\eta}{\Lambda}\Bigg] \\ & + \mathcal{O}(q^4)\ , \end{aligned} \quad (3.1)$$

where the state– and quark-mass–dependent coefficients $\gamma_B^{D,F}$ and α_B^P can be found, *e.g.*, in [4]. As announced, the baryon mass and the dimension-two couplings are renormalized as symbolized by the superscript (r). The precise renormalization takes the form

$$\begin{aligned} m_0^{(r,3)} &= m_0 - \left(\frac{5D^2}{36\pi^2F_0^2} + \frac{F^2}{4\pi^2F_0^2}\right)\Lambda^3\ , \\ b_F^{(r,3)} &= b_F - \left(\frac{5DF}{48F_0^2\pi^2}\right)\Lambda, \\ b_0^{(r,3)} &= b_0 - \left(\frac{13D^2+9F^2}{144\pi^2F_0^2}\right)\Lambda\ , \\ b_D^{(r,3)} &= b_D - \left(\frac{-D^2+3F^2}{32\pi^2F_0^2}\right)\Lambda. \end{aligned} \quad (3.2)$$

It is instructive to expand the $M_P^3 \arctan(M_P/\Lambda)$ ($P = \{\pi, K, \eta\}$) contributions

$$\begin{aligned} m_B = {} & m_0^{(r)} + \gamma_B^D b_D^{(r)} + \gamma_B^F b_F^{(r)} - 2b_0^{(r)}(M_\pi^2+2M_K^2) \\ & - \frac{1}{24\pi F_0^2}\left[\alpha_B^\pi M_\pi^3 + \alpha_B^K M_K^3 + \alpha_B^\eta M_\eta^3\right] \\ & + \frac{1}{12\pi^2F_0^2}\alpha_B^\pi M_\pi^3\left\{\frac{M_\pi}{\Lambda} - \frac{1}{3}\left(\frac{M_\pi}{\Lambda}\right)^3 + \ldots\right\} \\ & + \frac{1}{12\pi^2F_0^2}\alpha_B^K M_K^3\left\{\frac{M_K}{\Lambda} - \frac{1}{3}\left(\frac{M_K}{\Lambda}\right)^3 + \ldots\right\} \\ & + \frac{1}{12\pi^2F_0^2}\alpha_B^\eta M_\eta^3\left\{\frac{M_\eta}{\Lambda} - \frac{1}{3}\left(\frac{M_\eta}{\Lambda}\right)^3 + \ldots\right\} = \\ & m_0^{(r)} + \gamma_B^D b_D^{(r)} + \gamma_B^F b_F^{(r)} - 2b_0^{(r)}(M_\pi^2+2M_K^2) \\ & - \frac{1}{24\pi F_0^2}\left[\alpha_B^\pi M_\pi^3 + \alpha_B^K M_K^3 + \alpha_B^\eta M_\eta^3\right] + O(q^4)\ , \end{aligned} \quad (3.3)$$

where the last line corresponds to the DR result. As stated earlier, the additional contributions scale with inverse powers of the cut-off and thus vanish when $\Lambda \to \infty$ (which formally corresponds to DR). As already stressed in [20] (and noted by others), this third-order representation is not sufficiently accurate to make connection with lattice results if one is not very close to the physical value of the GB masses. Therefore, one should perform a fourth-order calculation or at least add an improvement term that is formally of fourth order but should be elevated to third order. We first discuss briefly this latter possibility before turning to the full-fledged fourth-order calculation.

3.2 Improvement term

As noticed in [20], the third-order result for the baryon masses shows a very strong cut-off dependence when the pion mass is increased above its physical value, see fig. 1. Only when one has a plateau below the chiral-symmetry-breaking scale $\Lambda_\chi = 4\pi F_\pi \simeq 1.2\,\mathrm{GeV}$, has one the required

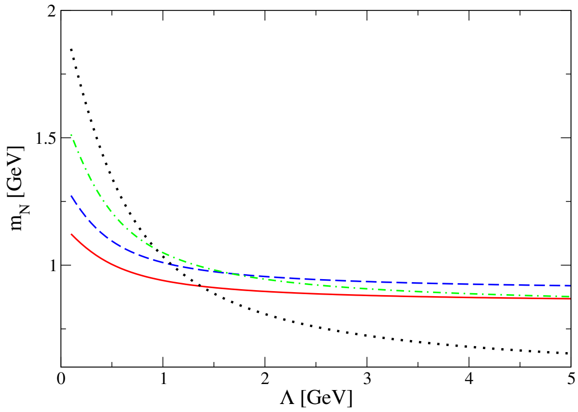

Fig. 1. Cut-off dependence of the nucleon mass for various pion masses with the kaon mass fixed at thrid order. Solid/dashed/dot-dashed/dotted line: $M_\pi = 140/300/450/600$ MeV.

cut-off independence. For pion masses above 300 MeV, this plateau vanishes.

To obtain a better stability against cut-off variations, it was therefore proposed in [20] to promote the fourth-order operator $e_1 M_\pi^4 \bar{N}N$ to the third order and to cancel the leading cut-off dependence in eq. (3.3) by a proper adjustment of the LEC e_1, $e_1 = e_1^{\rm fin} - {\rm coeff}/\Lambda$, where the coefficient can be read off from the leading term of the expansion of the arctan function. For the $SU(3)$ calculation performed here, the situation is a bit more complicated. In fact, the corresponding improvement term for the baryon B consists of three contributions (we use the notation of [15])

$$\epsilon_{1,B}^{\pi\pi} M_\pi^4 + \epsilon_{1,B}^{\pi K} M_\pi^2 M_K^2 + \epsilon_{1,B}^{KK} M_K^4 \ , \qquad (3.4)$$

where the coefficients $\epsilon_{1,B}^{PQ}$ $(P,Q=\{\pi,K\})$ are linear combinations of the fourth-order LECs d_i defined in eq. (2.7) (the precise relation can be found again in [15]). Throughout, we use the Gell-Mann–Okubo relation to express the M_η-term by the pion and the kaon masses, $3M_\eta^2 = 4M_K^2 - M_\pi^2$. To eliminate the leading $(1/\Lambda)$-dependences, the LECs d_i have to take the form

$$\begin{aligned}
d_1 &= d_1^{\rm fin} - \frac{D^2-3F^2}{576\pi^2 F_0^2 \Lambda} \ , \quad d_2 = d_2^{\rm fin} - \frac{DF}{64\pi^2 F_0^2 \Lambda} \ , \\
d_3 &= d_3^{\rm fin} - \frac{D^2-3F^2}{128\pi^2 F_0^2 \Lambda} \ , \quad d_4 = d_4^{\rm fin} - \frac{-D^2+3F^2}{64\pi^2 F_0^2 \Lambda} \ , \\
d_5 &= d_5^{\rm fin} + \frac{13DF}{288\pi^2 F_0^2 \Lambda} \ , \quad d_7 = d_7^{\rm fin} + \frac{35D^2+27F^2}{3456\pi^2 F_0^2 \Lambda} \ , \\
d_8 &= d_8^{\rm fin} + \frac{17D^2+9F^2}{1152\pi^2 F_0^2 \Lambda} \ . \qquad (3.5)
\end{aligned}$$

The $d_i^{\rm fin}$ are later identified with the finite pieces of the dimension-four LECs (after the renormalization at fourth order).

3.3 Baryon masses at fourth order

To fourth order in the chiral expansion, the octet baryon masses can be written as (when employing CR)

$$\begin{aligned}
m_B &= m_0^{(r)} + \delta m_B^{(2)} + \delta m_B^{(3)} + \delta m_B^{(4)} \\
&= m_0^{(r)} + \delta m_B^{(2)} + \Delta m_B^{(3)} + f_{B,3}(\Lambda) \\
&\quad + \Delta m_B^{(4)} + f_{B,4}(\Lambda) \ , \qquad (3.6)
\end{aligned}$$

where in the second line we have split the mass shift $\delta m_B^{(i)}$ $(i=3,4)$ into cut-off–independent and an explicitly cut-off–dependent piece. This is done to facilitate the comparison with the results obtained in DR. The various pieces take the form

$$\begin{aligned}
\delta m_B^{(2)} &= \gamma_B^D b_D^{(r)} + \gamma_B^F b_F^{(r)} - 2b_0^{(r)}(M_{0,\pi}^2 + 2M_{0,K}^2) \ , \\
\Delta m_B^{(3)} &= -\frac{1}{24\pi F_0^2}\left[\alpha_B^\pi M_\pi^3 + \alpha_B^K M_K^3 + \alpha_B^\eta M_\eta^3\right], \\
f_{B,3}(\Lambda) &= \frac{1}{12\pi^2 F_0^2}\left[\alpha_B^\pi M_\pi^3 \arctan\frac{M_\pi}{\Lambda}\right. \\
&\quad \left. + \alpha_B^K M_K^3 \arctan\frac{M_K}{\Lambda} + \alpha_B^\eta M_\eta^3 \arctan\frac{M_\eta}{\Lambda}\right] \ , \\
\Delta m_B^{(4)} &= \epsilon_{1,B}^{P,Q} M_P^2 M_Q^2 + \epsilon_{2,B,T}^{P,Q} M_P^2 M_Q^2 \ln\frac{M_T}{m_0} \ , \\
f_{B,4}(\Lambda) &= -\epsilon_{2,B,T}^{P,Q} M_P^2 M_Q^2 \ln(1+R_T) \\
&\quad + \frac{1}{R_T}\left\{\beta_{1,B,T}\Lambda^4\left(1 - R_T + \frac{1}{2}\left(\frac{M_T}{\Lambda}\right)^2\right)\right. \\
&\quad \left. + \beta_{2,B,T}^P M_P^2 \Lambda^2 (1-R_T) + \beta_{3,B,T}^{P,Q} M_P^2 M_Q^2\right\} \ , \\
& \qquad (3.7)
\end{aligned}$$

with $R_T = (1 + M_T^2/\Lambda^2)^{1/2}$. Here, we have $P,Q,T = \{\pi,K,\eta\}$ and the summation convention for these indices is understood. To arrive at these results, we made use of the loop integrals collected in appendix B. The resulting Λ-dependent terms are separated into a contribution that contains all the power and logarithmic divergences and another one that is finite in the limit that $\Lambda \to \infty$. Only these latter terms are displayed here. For the nucleon, the ϵ-coefficients and the β-coefficients are collected in appendix A. Note also that the fourth-order LECs d_i have the form displayed in eq. (3.5) so that the leading cut-off dependence is canceled. In the second-order term, we have used the leading terms in the quark mass expansion of the pion and the kaon mass, denoted $M_{0,P}$, because the difference to the physical masses appears in the fourth order of the baryon mass expansions. For consistency, we have recalculated the Goldstone boson masses in cut-off regularization (this was not done, *e.g.*, in [20]); the explicit formulae are given in appendix C. All appearing polynomial and logarithmic divergences are absorbed in a consistent redefinition of the bare parameters. The renormalization of the chiral-limit mass and of the LECs takes the form

$$
\begin{aligned}
m_0^{(r,4)} &= m_0^{(r,3)} + \frac{m_0}{\pi^2 F_0^2}\Big(-\frac{b_{12}}{8} - \frac{3b_4}{4} - \frac{7b_7}{12} - b_9 \\
&+ \frac{3b_{15}m_0}{4} + \frac{7b_{17}m_0}{12} + \frac{b_{18}m_0}{8} + b_{19}m_0 \Big) \Lambda^4, \\
b_F^{(r,4)} &= b_F^{(r,3)} - \frac{1}{96\pi^2 F_0^2}\Big(-10b_2 + 14b_F + 10b_F D^2 \\
&+ 20b_D DF + 18b_F F^2 - 5b_5 m_0 - 5b_6 m_0 \\
&+ 5b_{16}m_0^2 \Big) \Lambda^2, \\
b_0^{(r,4)} &= b_0^{(r,3)} - \frac{1}{144\pi^2 F_0^2}\Big(-18b_1 - 26b_3 - 48b_8 + 18b_D \\
&+ 48b_0 - 26b_D D^2 - 36b_F DF - 18b_D F^2 \\
&- 9b_4 m_0 - 13b_7 m_0 - 24b_9 m_0 + 9b_{15}m_0^2 \\
&+ 13b_{17}m_0^2 + 24b_{19}m_0^2 \Big) \Lambda^2, \\
b_D^{(r,4)} &= b_D^{(r,3)} - \frac{1}{96\pi^2 F_0^2}\Big(-18b_1 - 2b_3 + 14b_D \\
&+ 26b_D D^2 + 36b_F DF + 18b_D F^2 - 3b_{12}m_0 - 9b_4 m_0 \\
&- b_7 m_0 + 9b_{15}m_0^2 + b_{17}m_0^2 + 3b_{18}m_0^2 \Big) \Lambda^2, \\
d_1^{(r,4)} &= d_1 - \frac{1}{1152\pi^2 F_0^2 m_0}\Big(-D^2 + 3F^2 + 12b_1 m_0 \\
&- 4b_3 m_0 - 14b_D m_0 - 69b_D D^2 m_0 - 162b_F DF m_0 \\
&- 81b_D F^2 m_0 + 3b_4 m_0^2 - b_7 m_0^2 - 3b_{15}m_0^3 \\
&+ b_{17}m_0^3 \Big) \ln\frac{\Lambda}{m_0}, \\
d_2^{(r,4)} &= d_2 - \frac{1}{768\pi^2 F_0^2 m_0}\Big(-6DF - 12b_2 m_0 - 4b_F m_0 \\
&- 60b_F D^2 m_0 - 120b_D DF m_0 - 108b_F F^2 m_0 \\
&- 3b_5 m_0^2 - 3b_6 m_0^2 + 3b_{16}m_0^3 \Big) \ln\frac{\Lambda}{m_0}, \\
d_3^{(r,4)} &= d_3 - \frac{1}{768\pi^2 F_0^2 m_0}(-3D^2 + 9F^2 + 36b_1 m_0 \\
&+ 4b_3 m_0 - 24b_D m_0 - 78b_D D^2 m_0 - 108b_F DF m_0 \\
&- 54b_D F^2 m_0 + 3b_{12}m_0^2 + 9b_4 m_0^2 + b_7 m_0^2 \\
&- 9b_{15}m_0^3 - b_{17}m_0^3 - 3b_{18}m_0^3 \Big) \ln\frac{\Lambda}{m_0}, \\
d_4^{(r,4)} &= d_4 - \frac{1}{1152\pi^2 F_0^2 m_0}\Big(9D^2 - 27F^2 - 108b_1 m_0 \\
&+ 4b_3 m_0 + 44b_D m_0 + 288b_D D^2 m_0 - 6b_{12}m_0^2 \\
&- 27b_4 m_0^2 + b_7 m_0^2 + 27b_{15}m_0^3 - b_{17}m_0^3 \\
&+ 6b_{18}m_0^3 \Big) \ln\frac{\Lambda}{m_0}, \\
d_5^{(r,4)} &= d_5 - \frac{1}{1152\pi^2 F_0^2 m_0}\Big(26DF + 52b_2 m_0 - 44b_F m_0 \\
&+ 13b_5 m_0^2 + 13b_6 m_0^2 - 13b_{16}m_0^3 \Big) \ln\frac{\Lambda}{m_0}, \\
d_7^{(r,4)} &= d_7 - \frac{1}{6912\pi^2 F_0^2 m_0}\Big(35D^2 + 27F^2 + 108b_1 m_0 \\
&+ 140b_3 m_0 + 264b_8 m_0 - 132b_D m_0 - 264b_0 m_0 \\
&+ 144b_D D^2 m_0 + 27b_4 m_0^2 + 35b_7 m_0^2 + 66b_9 m_0^2 \\
&- 27b_{15}m_0^3 - 35b_{17}m_0^3 - 66b_{19}m_0^3 \Big) \ln\frac{\Lambda}{m_0}, \\
d_8^{(r,4)} &= d_8 - \frac{1}{2304\pi^2 F_0^2 m_0}\Big(17D^2 + 9F^2 + 36b_1 m_0 \\
&+ 68b_3 m_0 + 120b_8 m_0 - 28b_D m_0 - 120b_0 m_0 \\
&+ 168b_D D^2 m_0 + 432b_F DF m_0 + 216b_D F^2 m_0 \\
&+ 9b_4 m_0^2 + 17b_7 m_0^2 + 30b_9 m_0^2 - 9b_{15}m_0^3 \\
&- 17b_{17}m_0^3 - 30b_{19}m_0^3 \Big) \ln\frac{\Lambda}{m_0}\ . \qquad (3.8)
\end{aligned}
$$

Clearly, higher-order powers in the cut-off Λ appear as compared to the third-order calculation. Note also the appearance of a new scale in the logarithmically divergent terms. As can be seen from appendix B, the integrals I_1–I_6 and α_1–α_3 contain terms proportional to $\ln(M_P/\Lambda)$. To properly absorb these divergences, one uses $\ln(M_P/\Lambda) = \ln(M_P/\nu) - \ln(\Lambda/\nu)$, with ν the new scale. From here on, we set $\nu = m_0$. As a check, it can be shown that for $\Lambda \to \infty$ our fourth-order results agree with the ones obtained in DR in [15] if one sets $\mu = m_0$, with μ the scale of DR. For this comparison to work, one also has to account for the fact that in [15] some terms $\sim M_P^2 M_Q^2$ were absorbed in a redefinition of the d_i. Further, we remark that the addition of explicit decuplet degrees of freedom would not dramatically alter any of the conclusions drawn in our analysis. There is one topic, however, that needs to be discussed. For simplicity, we consider the $SU(2)$ case and just talk about the Δ-resonance and use DR with a scale λ. A theory without explicit delta degrees of freedom can be obtained from the one including explicit deltas expanding the result in the small quantity M_π/Δ_0, with Δ_0 the $N\Delta$ splitting in the chiral limit. In this limit logarithmic terms $\sim \log(M_\pi/2\Delta_0)$ originating from the $\Delta\pi$ branch cut obviously remain. However, such terms do not pose a problem for the chiral expansion in the theory without explicit deltas as one can show that the quark-mass–dependent structures $\log(M_\pi/\lambda)$ are consistent with the chiral expansion, whereas the structures $\sim \log(\lambda/2\Delta_0)$ get absorbed in counterterms and thus there are no new quark-mass–dependent structures. This concludes the necessary formalism, we now turn to the numerical analysis.

4 Results and discussion

Before presenting results, we must fix parameters. In the meson sector, we use standard values of the LECs L_i at the scale M_ρ: $L_4 = -0.3, L_5 = 1.4, L_6 = -0.2, L_7 = -0.4$ and $L_8 = 0.9$ (all in units of 10^{-3}). These are run to the scale $\lambda = m_0$ using the standard one-loop β-functions. Throughout, we set $F_0 = 100$ MeV, which is an average value of the physical values of F_π, F_K and F_η. For the leading baryon axial couplings we use $D = 0.75$ and $F = 0.5$. We have checked that varying these parameters

Table 1. Values of the LECs b_i in GeV^{-1} taken from [15].

b_0	b_D	b_F	b_1	b_2	b_3	b_8
-0.606	0.079	-0.316	-0.004	-0.187	0.018	-0.109

within phenomenological bounds does not alter our conclusions. We also stress that our main results are based on the fourth-order representation of the baryon masses, and we thus only present a detailed analysis of the theoretical uncertainties for that order. Also, the issue of the convergence of the chiral expansion can only be discussed when the complete fourth order is included.

4.1 Fixing the low-energy constants

For the dimension-two LECs from the meson-baryon Lagrangian we use the central values of [15]; these are collected in table 1 (when using CR, we have of course used the properly transformed values of these LECs). We have not varied these LECs since such modifications can effectively be done by changing the fourth-order LECs d_i within reasonable bounds. Next we consider the determination of the fourth-order LECs d_i. As noted before, the improvement term for each baryon consists of three pieces. To determine these, we have only considered the nucleon, for two reasons. First, there are more lattice results for this particle than for the others and second, it also facilitates the direct comparison with the $SU(2)$ results of [20]. For the nucleon, the coefficients appearing in eq. (3.4) are related to the LECs d_i via

$$\begin{aligned}
\epsilon_{1,N}^{\pi\pi} &= -4(4d_1+2d_5+d_7+3d_8)\ , \\
\epsilon_{1,N}^{\pi K} &= 8(4d_1-2d_2-d_5-2d_7+2d_8)\ , \\
\epsilon_{1,N}^{KK} &= -16(d_1-d_2+d_3-d_5+d_7+d_8)\ . \qquad (4.1)
\end{aligned}$$

We have now varied the values of the d_i under the following constraints: We require that $m_N(M_\pi)$ passes through the physical value $m_N = 940\,\mathrm{MeV}$ for the physical pion mass $M_\pi = 140\,\mathrm{MeV}$ and similarly for $m_N(M_K)$ at $M_K = 494\,\mathrm{MeV}$. Furthermore, we allow for variations of $\delta d_i = \pm 0.1\,\mathrm{GeV}^{-3}$ from the central values of [15] (this is in fact the largest magnitude of any of these LECs). Under these restrictions, we have tried to describe the trend of the earlier MILC data with the third-order improved formula (note that the more recent data [3] were not used in the fit for reasons discussed below). We have reconstructed these data from table VIII (VII) of ref. [2]([3]) using the scale parameter $r_1 = 0.35(0.317)\,\mathrm{fm}$. The resulting set of values for the LECs d_i is denoted as the "optimal set" from here on. In table 2 we have collected the values of the d_i from [15] and for the optimal set. In fact, the values of the d_i given in that table refer to the basis used in [15] as indicated by the superscript "BM". These differ by some small finite shifts from the one used in CR (we refrain from giving the explicit formulae here). Note that one cannot exactly reproduce these lattice data as shown by the solid line in fig. 2, but the nucleon mass

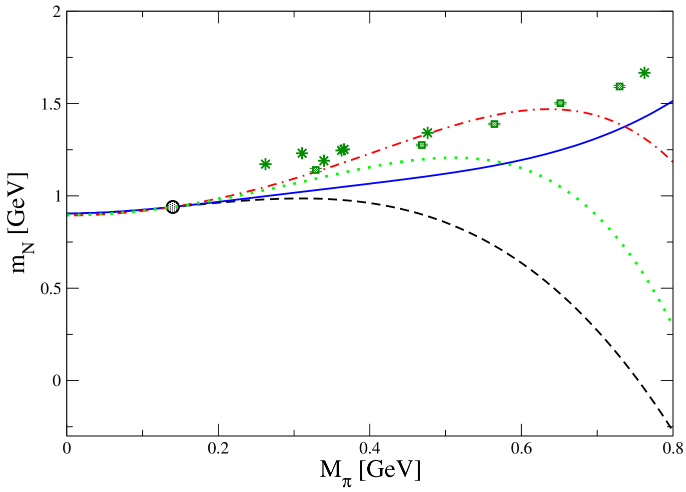

Fig. 2. Nucleon mass in DR at third (dashed), improved third (solid) and fourth (dot-dashed) order, respectively. The dotted line represents the fourth-order calculation from [15]. The three flavor data are from the MILC Collaboration (boxes from [2] and stars from [3]). The filled circle gives the value of the physical nucleon mass at the physical value of M_π.

now increases with growing pion mass as demanded by all existing lattice results. Also, at fourth order there are other contributions which are not captured by the improvement term, see the discussion below. We also note that we have not restricted the d_i such that the GMO relation $3m_\Lambda + m_\Sigma = 2(m_N + m_\Xi)$ is fulfilled so as to have a better handle on the theoretical uncertainty. Including the improvement term with these values of the LECs leads indeed to a very reduced cut-off dependence. Note that the treatment of the improvement term is more tricky in $SU(3)$ than in the $SU(2)$ calculation since one has to balance three different terms as opposed to fixing one in the two-flavor case.

4.2 Nucleon mass and pion-nucleon sigma term

We now consider the nucleon mass as a function of the pion and the kaon mass in DR and CR. We have calculated m_N at third, improved third and fourth order, see fig. 2 for DR. In what follows, we mostly focus on the results obtained at fourth order. The trends when going from third to improved third to fourth order for the pion mass dependence of the nucleon mass are very similar to the $SU(2)$ case discussed in great detail in [20]. We see that with the optimal set of the d_i as given in table 2 one obtains a rather accurate description of the earlier and most of the more recent MILC data [3] for pion masses below 600 MeV, cf. the dot-dashed line in fig. 2. Note, however, that the two lowest-pion-mass points of the recent MILC data [3] do not quite fit into the trend of our extrapolation function if one insists that for the physical pion mass the curve runs through the physical value of m_N. More low-mass pion data and/or a more sophisticated treatment of finite-size/volume effects are needed to resolve this

Table 2. Values of the LECs d_i in GeV^{-3} in the basis used in [15] as indicated by the superscript "BM". "Opt." denotes the optimal set as described in the text.

	$d_1^{\rm BM}$	$d_2^{\rm BM}$	$d_3^{\rm BM}$	$d_4^{\rm BM}$	$d_5^{\rm BM}$	$d_7^{\rm BM}$	$d_8^{\rm BM}$
BM [15]	0.008	0.035	0.069	−0.077	−0.05	−0.018	−0.103
Opt.	−0.043	−0.066	−0.031	−0.077	−0.15	−0.118	−0.2

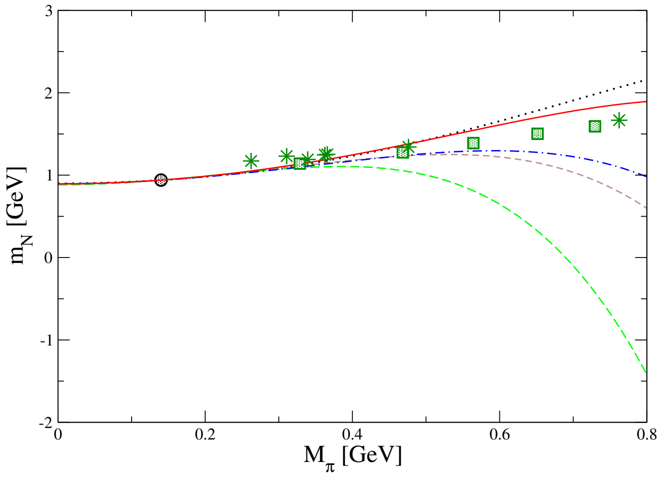

Fig. 3. Pion mass dependence of the nucleon mass in CR with $\Lambda = 1\,$GeV for various sets of the LECs d_i. The solid and the dot-dashed line correspond to the optimal set of the d_i and to the values from [15], respectively, and the various dashed lines are explained in the text.

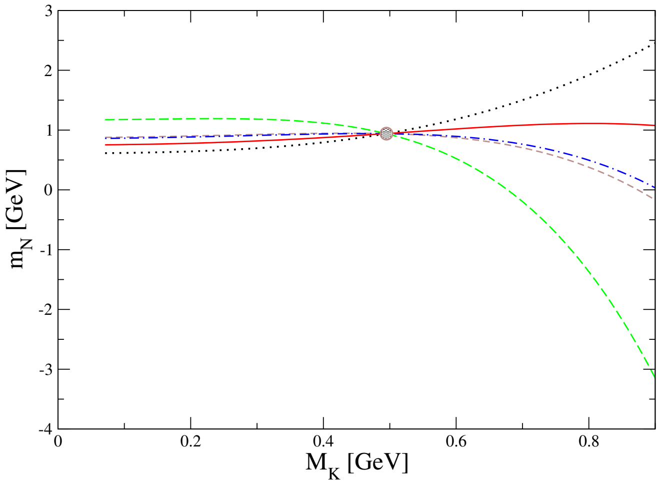

Fig. 4. Kaon mass dependence of the nucleon mass for various sets of the LECs d_i, see fig. 3.

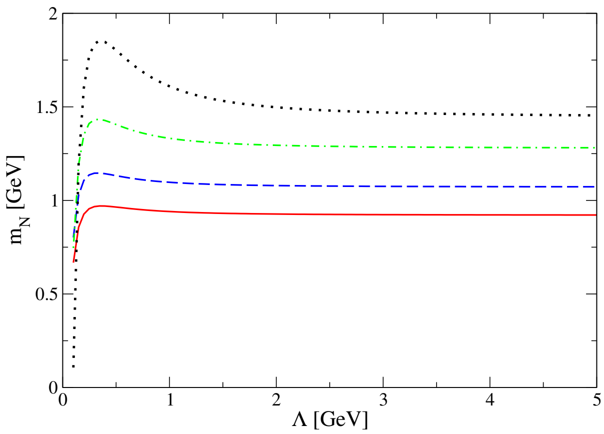

Fig. 5. Cut-off dependence of the nucleon mass for various pion masses with the kaon mass fixed. Solid/dashed/dot-dashed/dotted line: $M_\pi = 140/300/450/600\,$MeV.

problem. It was already stressed in [3] that these points are problematic concering a chiral extrapolation, and thus we did not include them in our fitting procedure. If one were to use the d_i determined in [15], one already deviates sizeably from the trend of the MILC data for pion masses starting at about 500 MeV (dotted line in fig. 2). The same can be seen for the fourth-order calculation based on CR utilizing $\Lambda = 1\,$GeV in fig. 3. To get a better idea about the uncertainty when going to higher pion masses, we have also performed calculations with three other sets, namely setting all $d_i = 0.2/0/-0.2\,$GeV^{-3}, corresponding to the long-/medium-/short-dashed lines in that figure. This clearly overestimates the theoretical uncertainty since some of the d_i are correlated parameters. Still, it is safe to say that for pion masses below 550 MeV the theoretical error is moderately small. These results for the pion mass dependence of m_N as well as for its cut-off dependence at a given pion mass are very similar to the results of the two-flavor study reported in [20]. In fig. 4, we show the kaon mass dependence for the same variety of choices for the d_i. Since we enforce that m_N takes its physical value for $M_K = 494\,$MeV, the resulting kaon mass dependence is much flatter than the pion mass dependence with decreasing meson masses. In fig. 5 we show the cut-off dependence of m_N for various values of the pion mass. For pion masses up to 450 MeV, one has a nice plateau below the scale of chiral symmetry breaking but not any more for $M_\pi = 600$ MeV. For the kaon mass dependence, the situation is somewhat different due to the much larger meson mass. Here, we still have a reasonable plateau at $M_K \simeq 600\,$MeV. These observations are consistent with our earlier observations that chiral extrapolations in M_π based on the fourth-order CHPT representation can be applied for masses up to 550 MeV, a result which is consistent with the one found for the $SU(2)$ calculation in [20]. For the kaon mass dependence, one can even go to somewhat higher meson masses.

It is also interesting to study the range of values found for the octet chiral-limit mass m_0 at the various orders and employing the different regularization schemes and

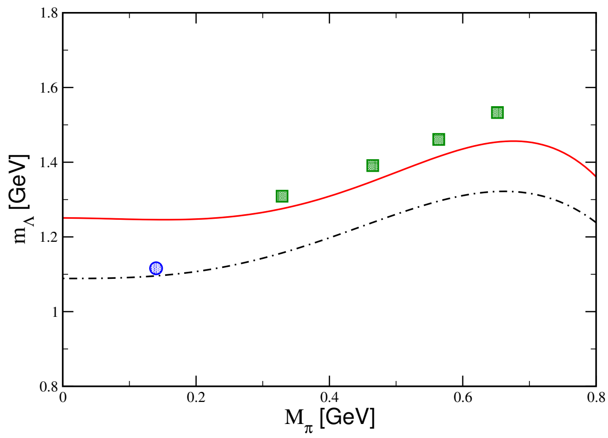

Fig. 6. Pion mass dependence of the Λ mass for various sets of the LECs d_i as explained in the text.

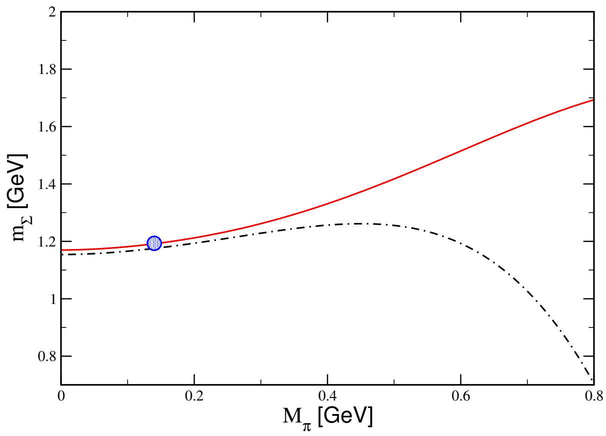

Fig. 8. Pion mass dependence of the Σ mass for various sets of the LECs d_i as explained in the text.

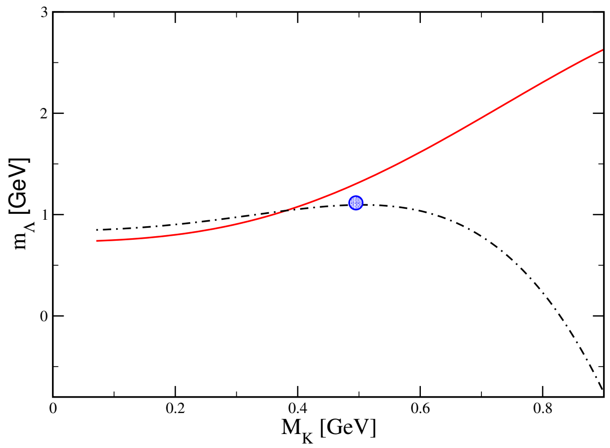

Fig. 7. Kaon mass dependence of the Λ mass for various sets of the LECs d_i as explained in the text.

values of the LECs d_i. One observes that m_0 increases with increasing cut-off, that means in DR its value is above the one in CR when one chooses $\Lambda = 1\,\text{GeV}$. Insisting that m_N takes its physical value at the physical value of M_π and M_K when studying the pion and the kaon mass dependence, respectively, we find

$$710 \text{ MeV} \lesssim m_0 \lesssim 1070 \text{ MeV} \ , \tag{4.2}$$

which is consistent with expectations and also with the findings in [15] (note that in that paper a different method was used to estimate the theoretical uncertainty, which we consider less reliable than the one used here). The range given in eq. (4.2) of course includes the $SU(2)$ value of about 880 MeV [20]. Also, we note again that the MILC data are obtained using staggered fermions, so strictly speaking one should use "staggered fermion CHPT". However, we believe that this will not significantly alter the trends discussed here.

Another quantity of interest is the pion-nucleon sigma term,

$$\sigma_{\pi N}(0) = \hat{m}\langle N|\bar{u}u + \bar{d}d|N\rangle = \hat{m}\frac{\partial m_N}{\partial \hat{m}} \ . \tag{4.3}$$

It can be extracted directly from the slope of the nucleon mass $m_N(M_\pi, M_K)$ at the physical value of the quark masses. For the optimal set and the LECs from [15], we obtain

$$\sigma_{\pi N}(0) = 44.9\ [39.5, 46.7]\ \text{MeV} \ , \tag{4.4}$$

where the values in the brackets are obtained from the variation of the d_i as discussed before (which is a conservative estimate of the uncertainty). These numbers are consistent with the results of [15] (as they should) and the study of $SU(2)$ lattice data in [23], $\sigma_{\pi N} = 49 \pm 3\,\text{MeV}$. The resulting strangeness fraction can be obtained from $\sigma_{\pi N}(0) = \sigma_0/(1-y)$ and using $\sigma_0 = 37\,\text{MeV}$ from [15] (which is consistent with the pioneering work in [24], $\sigma_0 = 35 \pm 5\,\text{MeV}$). This leads to $y = 0.07\ldots 0.22$, which is again consistent with [15] and also with the extraction of [25].

We end this section with some comments on the convergence of the chiral expansion. In fact, we have also performed calculations with sixth-order improvement terms (as defined in ref. [20]). None of our results for pion masses below 500 MeV was affected drastically by this. We therefore conclude, by comparing, *e.g.*, the dashed, solid and dot-dashed lines in fig. 2 and the various curves in fig. 3, that for pion masses less than 400 MeV the convergence is very good and that it is acceptable for masses below 550 MeV. This agrees with the findings in [20]. For the kaon mass dependence, our findings are very similar.

4.3 Hyperon masses

We now consider the octet members with strangeness. As noted before, when fixing the coefficients in the improvement term, we have not insisted to recover the Gell-Mann–Okubo relation, thus some of the masses are somewhat off their empirical values. In table 3 we collect the resulting values for the improved third and fourth order. While the Σ mass is well reproduced, the Λ and Ξ masses come out by about 10–15% too high. To get a handle on the theoretical accuracy, we also use the values for the d_i from [15]; in that case all masses are exactly reproduced.

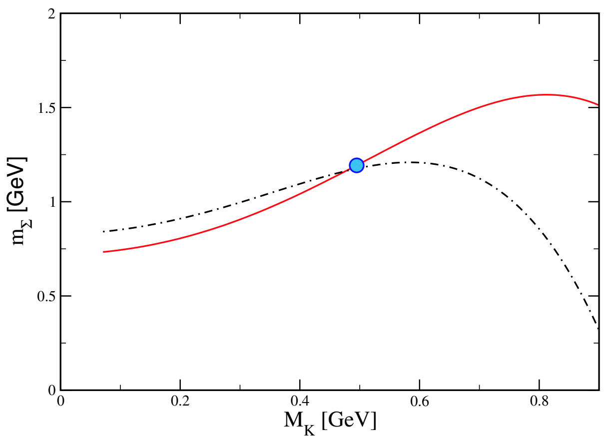

Fig. 9. Kaon mass dependence of the Σ mass for various sets of the LECs d_i as explained in the text.

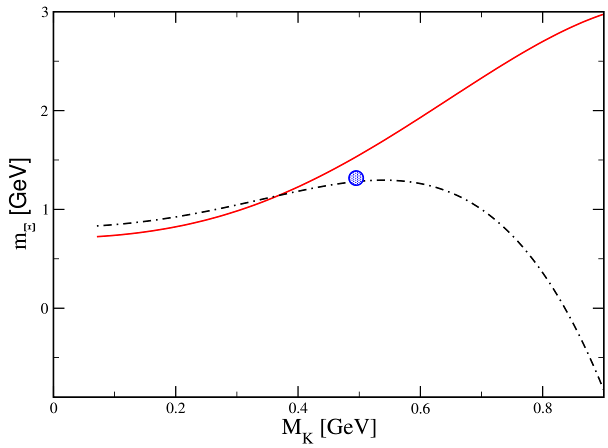

Fig. 11. Kaon mass dependence of the Ξ mass for various sets of the LECs d_i as explained in the text.

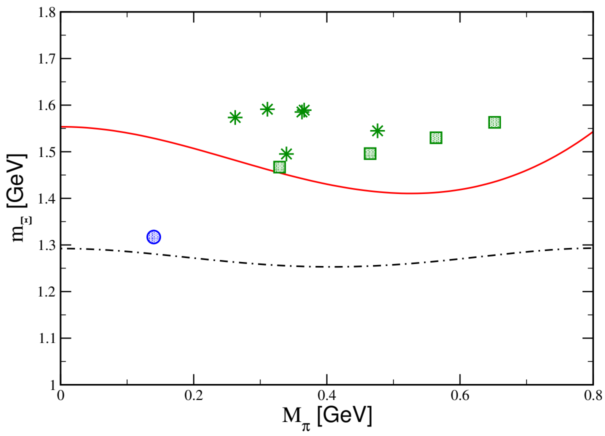

Fig. 10. Pion mass dependence of the Ξ mass for various sets of the LECs d_i as explained in the text.

Table 3. Baryon masses in MeV in DR and CR with $\Lambda = 1\,\text{GeV}$ for different orders. For the experimental numbers, we haven taken the masses of the neutral particles.

Order / baryon	Imp. 3rd CR	Fourth CR	Fourth DR	Exp.
Λ	1115	1304	1243	1116
Σ	1101	1194	1167	1193
Ξ	1222	1532	1437	1315

The corresponding pion and kaon mass dependences for the Λ, the Σ and the Ξ are shown in figs. 6-11, respectively. The dot-dashed lines refer to the optimal set of the LECs/to the LECs from [15]. We note in particular that the pion mass dependence for the Ξ is much flatter as one would expect from the MILC data. This is not unexpected —the Ξ only contains one valence light quark and should thus be less sensitive to variations in the pion mass. Clearly, one could improve this description by fitting directly to these particles.

We are grateful to Claude Bernard for some clarifying comments on the first version of this manuscript and Thomas Lippert for a very useful communication.

Appendix A. Baryon masses

Here, we collect some formalism to calculate the baryon masses from the baryon self-energies. Consider the heavy-baryon approach. The baryon four-momentum is $p_\mu = m_0 v_\mu + r_\mu$, with v_μ the four-velocity subject to the constraint $v^2 = 1$ and r_μ is the (small) residual four-momentum, $\omega = v \cdot r \ll m_0$. The baryon self-energy $\Sigma(\omega, r)$ has the chiral expansion

$$\Sigma(\omega, r) = \Sigma^{(2)}(\omega, r) + \Sigma^{(3)}(\omega, r) + \Sigma^{(4)}(\omega, r) + \dots\,, \tag{A.1}$$

and the corresponding baryon mass shift is then given by (with $\delta m_B = \delta m_B^{(2)} + \delta m_B^{(3)} + \delta m_B^{(4)} + \dots$):

$$\delta m_B^{(2)} = \Sigma^{(2)}(0, r) + \frac{r^2}{2m_0}, \tag{A.2}$$

$$\delta m_B^{(3)} = \Sigma^{(3)}(0, r) + \frac{\partial}{\partial\omega}\Sigma^{(2)}(0, r)\left(\delta m_B^{(2)} - \frac{r^2}{2m_0}\right), \tag{A.3}$$

$$\begin{aligned}\delta m_B^{(4)} = &-\frac{(\delta m_B^{(2)})^2}{2m_0} + \Sigma^{(4)}(0, r)\\ &+\frac{\partial}{\partial\omega}\Sigma^{(3)}(0, r)\left(\delta m_B^{(2)} - \frac{r^2}{2m_0}\right)\\ &+\frac{\partial}{\partial\omega}\Sigma^{(2)}(0, r)\,\delta m_B^{(3)}\\ &+\frac{1}{2}\frac{\partial^2}{\partial\omega^2}\Sigma^{(2)}(0, r)\left((\delta m_B^{(2)})^2 - \delta m_B^{(2)}\frac{r^2}{m_0} + \frac{r^4}{4m_0^2}\right).\end{aligned} \tag{A.4}$$

From this, one obtains the pertinent representations of the baryon masses. In the following, we only discuss the nucleon mass. More precisely, we now give the corresponding non-vanishing prefactors for the nucleon (we refrain from giving the coefficients of the other octet members). At third order, cf. eqs. (3.1), (3.3), one has the standard values

$$\begin{aligned}
\gamma_N^D &= -4M_K^2 \ , \quad \gamma_N^F = 4M_K^2 - 4M_\pi^2, \\
\alpha_N^\pi &= \frac{9}{4}(D+F)^2 \ , \quad \alpha_N^K = \frac{1}{2}(5D^2 - 6DF + 9F^2) \ , \\
\alpha_N^\eta &= \frac{1}{4}(D-3F)^2 \ .
\end{aligned} \tag{A.5}$$

At fourth order, see eq. (3.7), we have a cut-off–independent and a cut-off–dependent contribution to the nucleon mass shift. The coefficients of the Λ-independent term read

$$\begin{aligned}
\epsilon_{1,N}^{\pi\pi} &= -16d_1 - 8d_5 - 4d_7 - 12d_8, \\
\epsilon_{1,N}^{\pi K} &= 32d_1 - 16d_2 - 8d_5 - 16d_7 + 16d_8, \\
\epsilon_{1,N}^{KK} &= -16d_1 + 16d_2 - 16d_3 + 16d_5 - 16d_7 - 16d_8, \\
\epsilon_{2,N,\pi}^{\pi\pi} &= \frac{1}{2(4\pi F_0)^2}\bigg(- 12b_1 - 12b_2 - 12b_3 - 24b_8 + 12b_D \\
&\quad + 12b_F + 24b_0 - \frac{3D^2}{m_0} - \frac{6DF}{m_0} - \frac{3F^2}{m_0} \\
&\quad - 3b_4 m_0 - 3b_5 m_0 - 3b_6 m_0 - 3b_7 m_0 - 6b_9 m_0 \\
&\quad + 3b_{15} m_0^2 + 3b_{16} m_0^2 + 3b_{17} m_0^2 + 6b_{19} m_0^2\bigg), \\
\epsilon_{2,N,K}^{\pi K} &= \frac{1}{3(4\pi F_0)^2}\bigg(- 52b_D D^2 + 60b_F D^2 + 120b_D DF \\
&\quad - 72b_F DF - 36b_D F^2 + 108b_F F^2\bigg), \\
\epsilon_{2,N,K}^{KK} &= \frac{1}{3(4\pi F_0)^2}\bigg(- 36b_1 + 12b_2 - 36b_3 - 48b_8 + 36b_D \\
&\quad - 12b_F + 48b_0 + 52b_D D^2 - 60b_F D^2 - 120b_D DF \\
&\quad + 72b_F DF + 36b_D F^2 - 108b_F F^2 - \frac{5D^2}{m_0} + \frac{6DF}{m_0} \\
&\quad - \frac{9F^2}{m_0} - 3b_{12} m_0 - 9b_4 m_0 + 3b_5 m_0 + 3b_6 m_0 \\
&\quad - 9b_7 m_0 - 12b_9 m_0 + 9b_{15} m_0^2 - 3b_{16} m_0^2 + 9b_{17} m_0^2 \\
&\quad + 3b_{18} m_0^2 + 12b_{19} m_0^2\bigg). \\
\epsilon_{2,N,\eta}^{\pi\pi} &= \frac{1}{54(4\pi F_0)^2}\bigg(- 36b_1 + 12b_2 - 4b_3 - 24b_8 + 36b_D \\
&\quad - 60b_F + 24b_0 - \frac{D^2}{m_0} + \frac{6DF}{m_0} - \frac{9F^2}{m_0} - 9b_4 m_0 \\
&\quad + 3b_5 m_0 + 3b_6 m_0 - b_7 m_0 - 6b_9 m_0 + 9b_{15} m_0^2 \\
&\quad - 3b_{16} m_0^2 + b_{17} m_0^2 + 6b_{19} m_0^2\bigg), \\
\epsilon_{2,N,\eta}^{\pi K} &= \frac{1}{54(4\pi F_0)^2}\bigg(288b_1 - 96b_2 + 32b_3 + 192b_8 \\
&\quad - 240b_D + 336b_F - 192b_0 + \frac{8D^2}{m_0} - \frac{48DF}{m_0} \\
&\quad + \frac{72F^2}{m_0} + 72b_4 m_0 - 24b_5 m_0 - 24b_6 m_0 + 8b_7 m_0 \\
&\quad + 48b_9 m_0 - 72b_{15} m_0^2 + 24b_{16} m_0^2 \\
&\quad - 8b_{17} m_0^2 - 48b_{19} m_0^2\bigg), \\
\epsilon_{2,N,\eta}^{KK} &= \frac{1}{54(4\pi F_0)^2}\bigg(- 576b_1 + 192b_2 - 64b_3 \\
&\quad - 384b_8 + 384b_D - 384b_F + 384b_0 - \frac{16D^2}{m_0} \\
&\quad + \frac{96DF}{m_0} - \frac{144F^2}{m_0} - 144b_4 m_0 + 48b_5 m_0 + 48b_6 m_0 \\
&\quad - 16b_7 m_0 - 96b_9 m_0 + 144b_{15} m_0^2 \\
&\quad - 48b_{16} m_0^2 + 16b_{17} m_0^2 + 96b_{19} m_0^2\bigg) \ .
\end{aligned} \tag{A.6}$$

The corresponding coefficients of the Λ-dependent term are

$$\begin{aligned}
\beta_{1,N,\pi} &= \frac{m_0}{(4\pi F_0)^2}\bigg(- 3b_4 - 3b_5 - 3b_6 - 3b_7 \\
&\quad - 6b_9 + m_0(3b_{15} + 3b_{16} + 3b_{17} + 6b_{19})\bigg), \\
\beta_{1,N,K} &= \frac{m_0}{8(\pi F_0)^2}\bigg(- b_{12} - 3b_4 + b_5 + b_6 - 3b_7 \\
&\quad - 4b_9 + m_0(3b_{15} - b_{16} + 3b_{17} + b_{18} + 4b_{19})\bigg), \\
\beta_{1,N,\eta} &= \frac{m_0}{3(4\pi F_0)^2}\bigg(- 9b_4 + 3b_5 + 3b_6 - b_7 - 6b_9 \\
&\quad + m_0(9b_{15} - 3b_{16} + b_{17} + 6b_{19})\bigg), \\
\beta_{2,N,\pi}^{\pi} &= \frac{1}{(4\pi F_0)^2}\bigg(- 6b_1 - 6b_2 - 6b_3 - 12b_8 \\
&\quad + 6b_D + 6b_F + 12b_0 - 3b_4 m_0 - 3b_5 m_0 \\
&\quad - 3b_6 m_0 - 3b_7 m_0 - 6b_9 m_0 + 3b_{15} m_0^2 + 3b_{16} m_0^2 \\
&\quad + 3b_{17} m_0^2 + 6b_{19} m_0^2\bigg), \\
\beta_{2,N,K}^{\pi} &= \frac{1}{9(4\pi F_0)^2}\bigg(- 52b_D D^2 + 60b_F D^2 + 120b_D DF \\
&\quad - 72b_F DF - 36b_D F^2 + 108b_F F^2\bigg),
\end{aligned}$$

$$\beta^{K}_{2,N,K} = \frac{1}{9(4\pi F_0)^2}\Big(-108b_1 + 36b_2 - 108b_3 - 144b_8 + 108b_D - 36b_F + 144b_0 + 52b_D D^2 - 60b_F D^2 - 120b_D DF + 72b_F DF + 36b_D F^2 - 108b_F F^2 - 18b_{12}m_0 - 54b_4 m_0 + 18b_5 m_0 + 18b_6 m_0 - 54b_7 m_0 - 72b_9 m_0 + 54b_{15}m_0^2 - 18b_{16}m_0^2 + 54b_{17}m_0^2 + 18b_{18}m_0^2 + 72b_{19}m_0^2\Big),$$

$$\beta^{\pi}_{2,N,\eta} = \frac{1}{9(4\pi F_0)^2}\Big(18b_1 - 6b_2 + 2b_3 + 12b_8 - 18b_D + 30b_F - 12b_0 + 9b_4 m_0 - 3b_5 m_0 - 3b_6 m_0 + b_7 m_0 + 6b_9 m_0 - 9b_{15}m_0^2 + 3b_{16}m_0^2 - b_{17}m_0^2 - 6b_{19}m_0^2\Big),$$

$$\beta^{K}_{2,N,\eta} = \frac{1}{9(4\pi F_0)^2}\Big(-72b_1 + 24b_2 - 8b_3 - 48b_8 + 48b_D - 48b_F + 48b_0 - 36b_4 m_0 + 12b_5 m_0 + 12b_6 m_0 - 4b_7 m_0 - 24b_9 m_0 + 36b_{15}m_0^2 - 12b_{16}m_0^2 + 4b_{17}m_0^2 + 24b_{19}m_0^2\Big),$$

$$\beta^{\pi\pi}_{3,N,\pi} = \frac{1}{2(4\pi F_0)^2}\Big(-12b_1 - 12b_2 - 12b_3 - 24b_8 + 12b_D + 12b_F + 24b_0 - \frac{3D^2}{m_0} - \frac{6DF}{m_0} - \frac{3F^2}{m_o} - 3b_4 m_0 - 3b_5 m_0 - 3b_6 m_0 - 3b_7 m_0 + 3b_{16}m_0^2 + 3b_{17}m_0^2 + 6b_{19}m_0^2\Big),$$

$$\beta^{\pi K}_{3,N,K} = \frac{1}{3(4\pi F_0)^2}\Big(-52b_D D^2 + 60b_F\ D^2 + 120b_D\ DF - 72b_F DF - 36b_D F^2 + 108b_F F^2\Big),$$

$$\beta^{KK}_{3,N,K} = \frac{1}{3(4\pi F_0)^2}\Big(-36b_1 + 12b_2 - 36b_3 - 48b_8 + 36b_D - 12b_F + 48b_0 + 52b_D D^2 - 60b_F D^2 - 120b_D DF + 72b_F DF + 36b_D F^2 - 108b_F F^2 - \frac{5D^2}{m_0} + \frac{6DF}{m_0} - \frac{9F^2}{m_0} - 3b_{12}m_0 - 9b_4 m_0 + 3b_5 m_0 + 3b_6 m_0 - 9b_7 m_0 - 12b_9 m_0 + 9b_{15}m_0^2 - 3b_{16}m_0^2 + 9b_{17}m_0^2 + 3b_{18}{m_0}^2 + 12b_{19}m_0^2\Big),$$

$$\beta^{\pi\pi}_{3,N,\eta} = \frac{1}{(4\pi F_0)^2}\Big(-\frac{2b_1}{3} + \frac{2b_2}{9} - \frac{2b_3}{27} - \frac{4b_8}{9} + \frac{2b_D}{3} - \frac{10b_F}{9} + \frac{4b_0}{9} - \frac{D^2}{54m_0} + \frac{DF}{9m_0} - \frac{F^2}{6m_0} - \frac{b4m_0}{6} + \frac{b_5 m_0}{18} + \frac{b_6 m_0}{18} - \frac{b_7 m_0}{54} - \frac{b_9 m_0}{9} + \frac{b_{15}m_0^2}{6} - \frac{b_{16}m_0^2}{18} + \frac{b_{17}m_0^2}{54} + \frac{b_{19}m_0^2}{9}\Big),$$

$$\beta^{\pi K}_{3,N,\eta} = \frac{1}{(4\pi F_0)^2}\Big(\frac{16b_1}{3} - \frac{16b_2}{9} + \frac{16b_3}{27} + \frac{32b_8}{9} - \frac{40b_D}{9} + \frac{56b_F}{9} - \frac{32b_0}{9} + \frac{4D^2}{27m_0} - \frac{8DF}{9m_0} + \frac{4F^2}{3m_0} + \frac{4b_4 m_0}{3} - \frac{4b_5 m_0}{9} - \frac{4b_6 m_0}{9} + \frac{4b_7 m_0}{27} + \frac{8b_9 m_0}{9} - \frac{4b_{15}m_0^2}{3} + \frac{4b_{16}m_0^2}{9} - \frac{4b_{17}m_0^2}{27} - \frac{8b_{19}m_0^2}{9}\Big),$$

$$\beta^{KK}_{3,N,\eta} = \frac{1}{(4\pi F_0)^2}\Big(-\frac{32b_1}{3} + \frac{32b_2}{9} - \frac{32b_3}{27} - \frac{64b_8}{9} + \frac{64b_D}{9} - \frac{64b_F}{9} + \frac{64b_0}{9} - \frac{8D^2}{27m_0} + \frac{16DF}{9m_0} - \frac{8F^2}{3m_0} - \frac{8b_4 m_0}{3} + \frac{8b_5 m_0}{9} + \frac{8b_6 m_0}{9} - \frac{8b_7 m_0}{27} - \frac{16b_9 m_0}{9} + \frac{8b_{15}m_0^2}{3} - \frac{8b_{16}m_0^2}{9} + \frac{8b_{17}m_0^2}{27} + \frac{16b_{19}m_0^2}{9}\Big)\ . \qquad \text{(A.7)}$$

Finally, we note that we use the values of the b_i as given in table 1 and set all other dimension two LECs to zero.

Appendix B. Loop integrals in cut-off regularization

In this appendix, we collect all integrals needed in the calculation of the baryon self-energy to fourth order utilizing cut-off regularization. Here, M is a generic symbol for the propagating Goldstone boson and we give the explicit representations only for the relevant case $M > |\omega|$:

$$
\begin{aligned}
&I_1(M,\omega) \doteq \\
&\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}\frac{i}{\omega - v\cdot k + i0^+}(S\cdot k)^2 \\
&= -i\frac{1}{16\pi^2}\Bigg\{\frac{\Lambda^3}{3} - (M^2-\omega^2)\Lambda + \left(M^2-\omega^2\right)^{\frac{3}{2}}\pi \\
&-\left(M^2-\omega^2\right)^{\frac{3}{2}}\arctan\left(\frac{\sqrt{M^2-\omega^2}}{\Lambda}\right) \\
&+\frac{\omega\Lambda^2}{2}\sqrt{1+\frac{M^2}{\Lambda^2}} + \left(\omega^3 - \frac{3}{2}\omega M^2\right) \\
&\times \ln\left(\frac{\Lambda}{M} + \sqrt{1+\frac{\Lambda^2}{M^2}}\right) \\
&-(M^2-\omega^2)^{\frac{3}{2}}\arctan\left(\sqrt{\frac{M^2}{\omega^2}-1}\sqrt{1+\frac{M^2}{\Lambda^2}}\right)\Bigg\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
I_1(M,0) &= -i\frac{1}{16\pi^2}\Bigg\{\frac{\Lambda^3}{3} - M^2\Lambda + M^3\frac{\pi}{2} \\
&-M^3\arctan\left(\frac{M}{\Lambda}\right)\Bigg\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
I_2(M,\omega) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}\frac{i}{\omega - v\cdot k + i0^+} \\
&\times (S\cdot k)^2(v\cdot k) \\
&= -i\frac{1}{16\pi^2}\Bigg\{\frac{1}{4}\Lambda^4\sqrt{1+\frac{M^2}{\Lambda^2}} - \frac{3}{8}M^2\Lambda^2\sqrt{1+\frac{M^2}{\Lambda^2}} \\
&+\frac{3}{8}M^4\ln\left(\frac{\Lambda}{M}+\sqrt{1+\frac{\Lambda^2}{M^2}}\right)\Bigg\} + \omega I_1(M,\omega)\ ,
\end{aligned}
$$

$$
\begin{aligned}
I_2(M,0) &= -i\frac{1}{16\pi^2}\Bigg\{\frac{1}{4}\Lambda^4\sqrt{1+\frac{M^2}{\Lambda^2}} \\
&-\frac{3}{8}M^2\Lambda^2\sqrt{1+\frac{M^2}{\Lambda^2}} + \frac{3}{8}M^4\left[\ln\left(\frac{\Lambda}{m_0}\right) - \ln\left(\frac{M}{m_0}\right)\right. \\
&\left.+\ln\left(1+\sqrt{1+\frac{M^2}{\Lambda^2}}\right)\right]\Bigg\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
I_3(M,\omega) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}\frac{i^2}{(\omega - v\cdot k + i0^+)^2} \\
&\times (S\cdot k)^2 = -i\frac{\mathrm{d}}{\mathrm{d}\omega}I_1(M,\omega)
\end{aligned}
$$

$$
\begin{aligned}
I_3(M,0) &= -\frac{1}{16\pi^2}\Bigg\{\frac{\Lambda^2}{2}\sqrt{1+\frac{M^2}{\Lambda^2}} + M^2\frac{1}{\sqrt{1+\frac{M^2}{\Lambda^2}}} \\
&-\frac{3}{2}M^2\left[\ln\left(\frac{\Lambda}{m_0}\right) - \ln\left(\frac{M}{m_0}\right)\right. \\
&\left.+\ln\left(1+\sqrt{1+\frac{M^2}{\Lambda^2}}\right)\right]\Bigg\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
I_4(M,\omega) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}\frac{i^2}{(\omega - v\cdot k + i0^+)^2} \\
&\times (S\cdot k)^2(v\cdot k) \\
&= -i\frac{\mathrm{d}}{\mathrm{d}\omega}I_2(M,\omega) \\
&= -iI_1(M,\omega) + \omega I_3(M,\omega)\ , \\
I_4(M,0) &= -iI_1(M,0)\ , \\
I_5(M,\omega) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}\frac{i^2}{(\omega - v\cdot k + i0^+)^2} \\
&\times (S\cdot k)^2(v\cdot k)^2 \\
&= -i\omega I_1(M,\omega) - iI_2(M,\omega) + \omega^2 I_3(M,\omega)\ , \\
I_5(M,0) &= -iI_2(M,0)\ , \\
I_6(M,\omega) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}\frac{i^2}{(\omega - v\cdot k + i0^+)^2} \\
&\times (S\cdot k)^2 k^2 \\
&= M^2 I_3(M,\omega)\ , \\
I_6(M,0) &= M^2 I_3(M,0)\ . \qquad \text{(B.1)}
\end{aligned}
$$

Similarly, for the calculation of the baryon tadpole and the meson masses in CR we need the following integrals:

$$
\begin{aligned}
\alpha_1(M) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+} \\
&= \frac{1}{2(2\pi)^2}\Bigg\{\Lambda^2\sqrt{1+\frac{M^2}{\Lambda^2}} \\
&-M^2\left[\ln\left(\frac{\Lambda}{m_0}\right) - \ln\left(\frac{M}{m_0}\right)\right. \\
&\left.+\ln\left(1+\sqrt{1+\frac{M^2}{\Lambda^2}}\right)\right]\Bigg\}\ , \\
\alpha_2(M) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}k^2 = M^2\alpha_1(M)\ , \\
\alpha_3(M) &\doteq \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0^+}k_0^2 \\
&= \frac{1}{(2\pi)^2}\frac{1}{4}\Lambda^4\sqrt{1+\frac{M^2}{\Lambda^2}} + \frac{1}{4}M^2\alpha_1(M)\ . \qquad \text{(B.2)}
\end{aligned}
$$

Appendix C. Meson masses in cut-off regularization

Here, we collect the formulaae for the meson masses to fourth order in CR. The pertinent diagrams are tree graphs with one insertion from $\mathcal{L}_\phi^{(2)}$ and $\mathcal{L}_\phi^{(4)}$ and tadpoles with exactly one insertion from $\mathcal{L}_\phi^{(2)}$. We have

$$
\begin{aligned}
M_\pi^2 &= M_{0,\pi}^2 + \delta M_\pi^{(4)} \\
&= M_{0,\pi}^2 + \frac{M_\pi^4}{16F_0^2\pi^2}\ln\frac{M_\pi}{m_0} - \frac{M_\pi^2 M_\eta^2}{48F_0^2\pi^2}\ln\frac{M_\eta}{m_0} \\
&- \frac{16L_4^{(r)}M_K^2M_\pi^2}{F_0^2} + \frac{32L_6^{(r)}M_K^2M_\pi^2}{F_0^2} - \frac{8L_4^{(r)}M_\pi^4}{F_0^2} \\
&- \frac{8L_5^{(r)}M_\pi^4}{F_0^2} + \frac{16L_6^{(r)}M_\pi^4}{F_0^2} + \frac{16L_8^{(r)}M_\pi^4}{F_0^2} \\
&- \frac{M_\pi^4}{16F_0^2\pi^2}\ln\left(1+\sqrt{1+\left(\frac{M_\pi}{\Lambda}\right)^2}\right) \\
&+ \frac{M_\pi^2M_\eta^2}{48F_0^2\pi^2}\ln\left(1+\sqrt{1+\left(\frac{M_\eta}{\Lambda}\right)^2}\right) \\
&+ \frac{1}{\sqrt{1+(\frac{M_\pi}{\Lambda})^2}}\left\{\frac{M_\pi^2}{16F_0^2\pi^2}\Lambda^2\left(1-\sqrt{1+\left(\frac{M_\pi}{\Lambda}\right)^2}\right)\right. \\
&\left. + \frac{M_\pi^4}{16F_0^2\pi^2}\right\} \\
&+ \frac{1}{\sqrt{1+(\frac{M_\eta}{\Lambda})^2}}\left\{-\frac{M_\pi^2}{48F_0^2\pi^2}\Lambda^2\left(1-\sqrt{1+\left(\frac{M_\eta}{\Lambda}\right)^2}\right)\right. \\
&\left. - \frac{M_\pi^2M_\eta^2}{48F_0^2\pi^2}\right\},
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
M_K^2 &= M_{0,K}^2 + \delta M_K^{(4)} \\
&= M_{0,K}^2 + \frac{M_K^2M_\eta^2}{24F_0^2\pi^2}\ln\frac{M_\eta}{m_0} + \frac{M_K^2M_\pi^2}{F_0^2}(-8L_4^{(r)}+16L_6^{(r)}) \\
&+ \frac{M_K^4}{F_0^2}(-16L_4^{(r)} - 8L_5^{(r)} + 32L_6^{(r)} + 16L_8^{(r)}) \\
&- \frac{M_K^2M_\eta^2}{24F_0^2\pi^2}\ln\left(1+\sqrt{1+\left(\frac{M_\eta}{\Lambda}\right)^2}\right) \\
&+ \frac{1}{\sqrt{1+(\frac{M_\eta}{\Lambda})^2}}\left\{\frac{M_K^2}{24F_0^2\pi^2}\Lambda^2\left(1-\sqrt{1+\left(\frac{M_\eta}{\Lambda}\right)^2}\right)\right. \\
&\left. - \frac{M_K^2M_\eta^2}{24F_0^2\pi^2}\right\},
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
M_\eta^2 &= M_{0,\eta}^2 + \delta M_\eta^{(4)} \\
&= M_{0,\eta}^2 - \frac{M_\pi^4}{16F_0^2\pi^2}\ln\frac{M_\pi}{m_0} + \frac{M_K^4}{6F_0^2\pi^2}\ln\frac{M_K}{m_0} \\
&+ \left(-\frac{7M_\pi^4}{432F_0^2\pi^2} + \frac{11M_K^2M_\pi^2}{108F_0^2\pi^2} - \frac{4M_K^4}{27F_0^2\pi^2}\right)\ln\frac{M_\eta}{m_0} \\
&+ \frac{M_K^4}{F_0^2}\left(-\frac{64L_4^{(r)}}{3} - \frac{128L_5^{(r)}}{9} + \frac{128L_6^{(r)}}{3}\right. \\
&\qquad \left. + \frac{128L_7^{(r)}}{3} + \frac{128L_8^{(r)}}{3}\right) \\
&+ \frac{M_K^2M_\pi^2}{F_0^2}\left(-\frac{16L_4^{(r)}}{3} + \frac{64L_5^{(r)}}{9} + \frac{32L_6^{(r)}}{3}\right. \\
&\qquad \left. - \frac{256L_7^{(r)}}{3} - \frac{128L_8^{(r)}}{3}\right) \\
&+ \frac{M_\pi^4}{F_0^2}\left(\frac{8L_4^{(r)}}{3} - \frac{8L_5^{(r)}}{9} - \frac{16L_6^{(r)}}{3}\right. \\
&\qquad \left. + \frac{128L_7^{(r)}}{3} + 16L_8^{(r)}\right) \\
&+ \frac{M_\pi^4}{16F_0^2\pi^2}\ln\left(1+\sqrt{1+\left(\frac{M_\pi}{\Lambda}\right)^2}\right) \\
&- \frac{M_K^4}{6F_0^2\pi^2}\ln\left(1+\sqrt{1+\left(\frac{M_K}{\Lambda}\right)^2}\right) \\
&- \left(-\frac{7M_\pi^4}{432F_0^2\pi^2} + \frac{11M_K^2M_\pi^2}{108F_0^2\pi^2} - \frac{4M_K^4}{27F_0^2\pi^2}\right) \\
&\qquad \times \ln\left(1+\sqrt{1+\left(\frac{M_\eta}{\Lambda}\right)^2}\right) \\
&+ \frac{1}{\sqrt{1+(\frac{M_\pi}{\Lambda})^2}} \\
&\times\left\{-\frac{M_\pi^2}{16F_0^2\pi^2}\Lambda^2\left(1-\sqrt{1+\left(\frac{M_\pi}{\Lambda}\right)^2}\right) - \frac{M_\pi^4}{16F_0^2\pi^2}\right\} \\
&+ \frac{1}{\sqrt{1+(\frac{M_K}{\Lambda})^2}} \\
&\times\left\{\frac{M_K^2}{6F_0^2\pi^2}\Lambda^2\left(1-\sqrt{1+\left(\frac{M_K}{\Lambda}\right)^2}\right) + \frac{M_K^4}{6F_0^2\pi^2}\right\} \\
&+ \frac{1}{\sqrt{1+(\frac{M_\eta}{\Lambda})^2}}\left\{\left(-\frac{M_K^2}{9F_0^2\pi^2} + \frac{7M_\pi^2}{144F_0^2\pi^2}\right)\right. \\
&\times\Lambda^2\left(1-\sqrt{1+\left(\frac{M_\eta}{\Lambda}\right)^2}\right) \\
&\left. - \frac{4M_K^2}{27F_0^2\pi^2} + \frac{11M_\pi^2M_K^2}{108F_0^2\pi^2} - \frac{7M_\pi^2}{432F_0^2\pi^2}\right\}.
\end{aligned}
$$

As required, in the limit $\Lambda \to \infty$ we recover the standard DR result [26]. The polynomial and logarithmic divergences in the cut-off are taken care of by the following renormalization (note again that, *e.g.*, B_0 is not renormalized in DR):

$$\begin{aligned}
B_0^{(r)} &= B_0 + \frac{1}{24\pi^2 F_0^2} B_0 \Lambda^2 \ , \\
L_7^{(r)} + \frac{L_8^{(r)}}{3} &= L_7 + \frac{L_8}{3} + \frac{5}{2304\pi^2} \ln \frac{\Lambda}{m_0} , \\
L_5^{(r)} - 2L_8^{(r)} &= L_5 - 2L_8 + \frac{1}{96\pi^2} \ln \frac{\Lambda}{m_0} \ , \\
L_4^{(r)} - 2L_6^{(r)} &= L_4 - 2L_6 - \frac{1}{576\pi^2} \ln \frac{\Lambda}{m_0} . \qquad \text{(C.3)}
\end{aligned}$$

Here, B_0 connects the leading terms in the chiral expansion of the Goldstone boson masses with the quark masses,

$$\begin{aligned}
M_{0,\pi}^2 &= 2B_0^{(r)} \hat{m} \ , \\
M_{0,K}^2 &= B_0^{(r)} (\hat{m} + m_s) \ , \\
M_{0,\eta}^2 &= \frac{2}{3} B_0^{(r)} (\hat{m} + 2m_s) \ , \qquad \text{(C.4)}
\end{aligned}$$

with $\hat{m}$ the average light quark mass.

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